MATH 211 EXAM (2024)

(3 hours)

No books, notes, or electronic devices (especially no phones, but also no smart watches, smart pens, headphones etc) are permitted during this exam.

You must show your work to receive credit. **Justify everything**.

Do not unstaple the exam or reorder the pages. All problems must be solved within the space provided (right after the statement of the problem). If you need to use the extra pages at the end, then mention this clearly in the aforementioned space, so your grader knows that they have to also look at the end (they will not check the extra pages unless explicitly told to).

We will provide scratch paper (loose sheets) but do not write solutions on them. **Only the 24 pages of the booklet you're now reading will be graded**.

Please do not leave the room during the first and last 30 minutes of the exam.

Please keep your CAMIPRO face up on the table at all times.

Don't forget to write your SCIPER and sign the exam.

There are 8 problems, worth 100 points in total.

SCIPER: ____________________

SIGNATURE: ____________________

PROBLEM 1

(a) Which of the following operations are associative, i.e. $(a*b)*c = a*(b*c)$ (justify: if they are associative include a proof, if they are not associative then provide a counterexample)?

- The operation $*$ on $\mathbb{R}$ defined by $a * b = \dfrac{a+b}{2}$. *(3 points)*

- The operation $*$ on the set of all finite words on a given alphabet S (i.e. arbitrary sequences $w_1 \dots w_k$ where $w_1, \dots, w_k \in S$ and $k \geq 0$ are arbitrary) given by concatenation, e.g. ginger $*$ bread $=$ gingerbread. *(3 points)*

(b) Consider the usual addition modulo 4 on the set $\mathbb{Z}/4\mathbb{Z}$. Show that this operation makes $\mathbb{Z}/4\mathbb{Z}$ into a group, by checking all the group axioms. *(3 points)*

(c) However, consider the usual multiplication modulo 4 on the set $\mathbb{Z}/4\mathbb{Z}$. Show that this operation does not make $\mathbb{Z}/4\mathbb{Z}$ into a group, by showing that at least one of the group axioms is violated. *(3 points)*

PROBLEM 2

(a) If H_1 and H_2 are subgroups of a group G, prove that $H_1 \cup H_2$ is a subgroup of G if and only if one of the subgroups is contained inside the other (i.e. $H_1 \subseteq H_2$ or $H_2 \subseteq H_1$).

(8 points)

(b) Let G be the symmetric group S_5, and let $x \in G$ be the 5-cycle $(1\ 2\ 3\ 4\ 5)$, i.e. the permutation taking $1 \to 2 \to 3 \to 4 \to 5 \to 1$. Show that the centralizer $C_G(x)$ (i.e. the set of all elements of G which commute with x) coincides with the subgroup $\langle x \rangle$ generated by x (i.e. the subset of G consisting of all integer powers of x). *(8 points)*

PROBLEM 3

Consider the action of the symmetric group S_3 on the 9-element set

$$X = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}$$

given by

$$\sigma \cdot (i,j) = (\sigma(i), \sigma(j))$$

for all $\sigma \in S_3$ and all $i, j \in \{1,2,3\}$ (you don't need to check that it's a well-defined action).

(a) Explicitly describe the orbits of the action $S_3 \curvearrowright X$ defined above. *(4 points)*

(b) Explicitly describe the stabilizers of elements in the orbits you found in part (a), and fill in the dots below with the correct numbers in the orbit-stabilizer theorem.

$$|X| = |\text{first orbit}| + |\text{second orbit}| = \frac{|S_3|}{\dots} + \frac{|S_3|}{\dots}$$

(it's enough to describe the stabilizer of a single element in any given orbit). *(6 points)*

PROBLEM 4

Consider the dihedral group D_{2n} with $n \geq 3$, and the sequence of homomorphisms

$$1 \to \mathbb{Z}/n\mathbb{Z} \xrightarrow{f} D_{2n} \xrightarrow{g} \mathbb{Z}/2\mathbb{Z} \to 1 \tag{1}$$

where $f(k \text{ mod } n) = (\text{rotation by } \frac{2\pi k}{n} \text{ radians})$ for all $k \in \mathbb{Z}$, while $g(\text{rotations}) = (0 \text{ mod } 2)$ and $g(\text{reflections}) = (1 \text{ mod } 2)$. You may assume that f and g are indeed homomorphisms.

(a) Check that (1) is a short exact sequence (i.e. check all the axioms). *(5 points)*

For parts (b) and (c), do not invoke any statement of the sort "D_{2n} is / isn't a (semi)direct product". Prove the statements using only direct computations and the definition of D_{2n}.

(b) Prove that there <u>exists</u> a homomorphism $\psi : \mathbb{Z}/2\mathbb{Z} \to D_{2n}$ such that $g \circ \psi = \mathrm{Id}_{\mathbb{Z}/2\mathbb{Z}}$.

(4 points)

(c) Prove that there does not exist a homomorphism $\phi : D_{2n} \to \mathbb{Z}/n\mathbb{Z}$ such that $\phi \circ f = \mathrm{Id}_{\mathbb{Z}/n\mathbb{Z}}$ (remember that we assumed $n \geq 3$). *(4 points)*

PROBLEM 5

(a) Prove that if G, G', H are finite abelian groups such that $G \times H$ is isomorphic to $G' \times H$, then G is isomorphic to G'. *(8 points)*

(b) Find a counterexample to part (a) if we drop the "finite abelian" assumption, i.e. find groups G, G', H such that $G \ncong G'$ but $G \times H \cong G' \times H$. Prove your assertions.

(4 points, hard)

PROBLEM 6

(a) If G_1, G_2, H_1, H_2 are simple groups such that there exists an isomorphism

$$G_1 \times G_2 \cong H_1 \times H_2$$

then prove that either

- $G_1 \cong H_1$ and $G_2 \cong H_2$, or
- $G_1 \cong H_2$ and $G_2 \cong H_1$.

(the two options are not exclusive, i.e. they are allowed to hold at the same time) (5 points)

(b) Show that the symmetric group S_4 is solvable, by exhibiting subgroups

$$\{1\} = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_{k-1} \triangleleft G_k = S_4$$

where G_{i-1} is normal inside G_i and the quotient G_i/G_{i-1} is a <u>cyclic</u> group, for all $i \in \{1, \ldots, k\}$. The choice of number k is up to you. *(10 points)*

PROBLEM 7

(a) Show that any group of order 56 has a normal Sylow p-subgroup for some prime $p \in \{2, 7\}$.

Partial credit will be given for useful remarks concerning the number of Sylow p-subgroups, or when such a subgroup is normal, which apply to the problem at hand. *(8 points)*

(b) Find, with proof, which well-known group is isomorphic to a Sylow 2-subgroup of S_5.

(4 points, hard)

PROBLEM 8

Show that a finite group G is nilpotent if and only if

$$xy = yx$$

for all $x, y \in G$ whose orders are coprime. *You may use the fact that G is nilpotent $\Leftrightarrow$ it is the direct product of its Sylow p-subgroups $\Leftrightarrow$ all the Sylow p-subgroups of G are normal.*

only if implication, i.e. G nilpotent implies "$xy = yx$, $\forall x, y$ of coprime orders". *(6 points)*

(see next page)

if implication, i.e. "$xy = yx, \forall x, y$ of coprime orders" implies G nilpotent. *(4 points, hard)*

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